Lecture 17. Mechanical Vibrations Part 2

Forced Oscillations

In this Lecture, we will talk about the systems with forced oscillations.

We have the differential equation

$$mx'' + cx' + kx = F(t)$$

with

$$F(t) = F_0 \cos \omega t \qquad \text{or} \qquad F(t) = F_0 \sin \omega t$$

where the constant F_0 is the amplitude of the periodic force and ω is its circular frequency.

Undamped Forced Oscillations

We set $c = 0$ and consider

$$mx'' + kx = F_0 \cos \omega t \tag{1}$$

Discussion:

- By the previous lectures, the complementary function is

$$x_c = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t,$$

where $w_0 = \sqrt{\frac{k}{m}} \Rightarrow k = w_0^2 m$.

- Assume $w_0 \neq w$. we want to find a particular solution x_p of Eq(1).
- Assume $x_p = A \cos \omega t, \quad x_p'' = -A\omega^2 \cos wt$ then

$$mx_p'' + kx_p = -Am\omega^2 \cos wt + kA \cos \omega t = F_0 \cos wt$$
$$\Rightarrow A\left(k - m\omega^2\right) = F_0 \Rightarrow A = \frac{F_0}{k - mw^2} = \frac{F_0/m}{\omega_0^2 - w^2},$$

the last equation is from the fact that $k = \omega_0^2 m$.

- Thus

$$x_p = \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t$$

- Therefore the general solution

$$x(t) = x_c(t) + x_p(t) = c_1 \cos w_0 t + c_2 \sin \omega_0 t + \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t$$
$$\Rightarrow x(t) = C \cos(\omega_0 t - \alpha) + \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t$$

- So $x(t)$ is a superposition of two oscillations.

Resonance

Recall on previous page, we have the particular solution of $mx'' + kx = F_0 \cos \omega t$ is given by

- $x_p(t) = \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t$, where $\omega_0 = \sqrt{\frac{k}{m}}$
- Resonance occurs when $\omega_0^2 = \omega^2$.
    - Roughly speaking, mechanical resonance is the phenomenon where a mechanical system vibrates with increased amplitude when the frequency of its oscillations matches the system's natural frequency.

Reading material on resonance:

- Mechanical resonance
- Tacoma Narrows Bridge

Example 1 Express the solution of the given initial value problem as a sum of two oscillations. Graph the solution function $x(t)$ in such a way that you can identify and label its period.

$$x'' + 25x = 9\cos 2t; \qquad x(0) = 0, \quad x'(0) = 0$$

ANS: Find x_c. $r^2+25=0 \Rightarrow r = \pm 5i$

$$x_c = C_1 \cos 5t + C_2 \sin 5t$$

Find x_p. Assume $x_p = A\cos 2t$, then $x_p'' = -4A\cos 2t$.

$x_p'' + 25x_p = 9\cos 2t \Rightarrow (-4A+25A)\cos 2t = 9\cos 2t$

$\Rightarrow 21A = 9 \Rightarrow A = \frac{3}{7}$

The general solution for nonhomogeneous eqn is

$$x(t) = x_c + x_p = C_1\cos 5t + C_2 \sin 5t + \frac{3}{7}\cos 2t$$

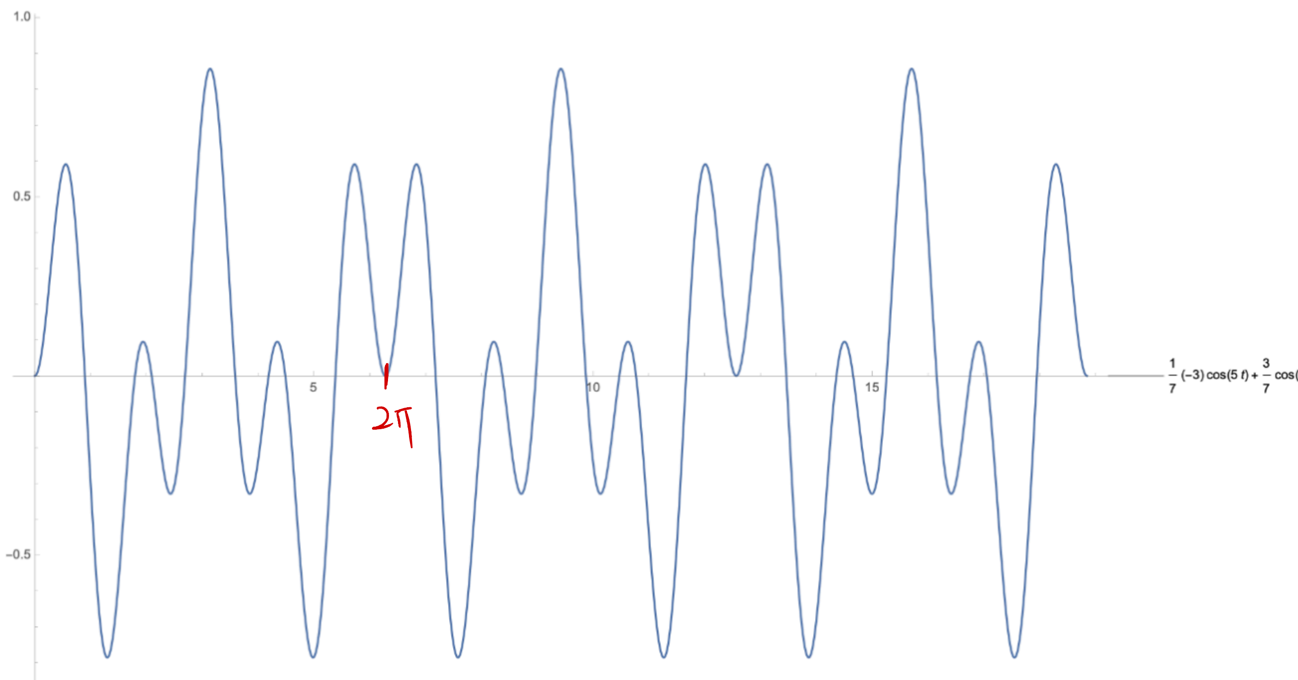
As $x(0)=0$, $x(0) = C_1 + \frac{3}{7} = 0 \Rightarrow C_1 = -\frac{3}{7}$

As $x'(0)=0$, $x'(t) = -5C_1 \sin 5t + 5C_2 \cos 5t - \frac{6}{7}\sin 2t$

$x'(0) = 5C_2 = 0 \Rightarrow C_2 = 0$.

Thus. $x(t) = -\frac{3}{7}\cos 5t + \frac{3}{7}\cos 2t$, which is a sum of two oscillations.

The period of $x(t)$ is the least common multiple of the periods of the two oscillations $\frac{2\pi}{5}$ and $\frac{2\pi}{2}$, which is 2π.



Damped Forced Oscillations

$$mx'' + cx' + kx = F_0 \cos \omega t$$

- **transient solution** $x_{\text{tr}}(t) = x_c(t), \quad x_c(t) \to 0$ as $t \to \infty$.
- **steady periodic solution** $x_{\text{sp}}(t) = x_p(t)$

Example 2. Consider the initial value problem

$$mx'' + cx' + kx = F(t), \quad x(0) = 0, \quad x'(0) = 0$$

modeling the motion of a spring-mass-dashpot system initially at rest and subjected to an applied force $F(t)$, where the unit of force is the Newton (N).

Assume that $m = 2$ kilograms, $c = 8$ kilograms per second, $k = 80$ Newtons per meter, and $F(t) = 20 \sin(6t)$ Newtons.

(1) Solve the initial value problem.

(2) Determine the long-term behavior of the system. Is $\lim\limits_{t \to \infty} x(t) = 0$?

ANS: (1) We have

$$2x'' + 8x' + 80x = 20 \sin 6t, \quad x(0) = 0, \quad x'(0) = 0.$$

The corresponding homogenous eqn is

$$2x'' + 8x' + 80x = 0 \Rightarrow x'' + 4x' + 40x = 0$$

The char. eqn is

$$r^2 + 4r + 40 = 0$$

$$\Rightarrow \quad r_{1,2} = \frac{-4 \pm \sqrt{16 - 4 \times 40}}{2} = -2 \pm 6i$$

Thus $\underline{x_c = e^{-2t}(C_1 \cos 6t + C_2 \sin 6t)}$

Next, we want to find x_p.

Assume $x_p(t) = A \cos 6t + B \sin 6t$.

Then $x_p'(t) = -6A\sin 6t + 6B\cos 6t$

$x_p''(t) = -36A\cos 6t - 36B\sin 6t$

Plug them into the given eqn, we have

$$\overset{4}{\cancel{2}}x_p'' + \overset{40}{\cancel{8}}x_p' + \cancel{80}x_p = \overset{10}{\cancel{20}}\sin 6t,$$

$$\Rightarrow \; -36A\cos 6t - 36B\sin 6t - 24A\sin 6t + 24B\cos 6t$$
$$+ 40A\cos 6t + 40B\sin 6t = 10\sin 6t$$

$$\Rightarrow \; (4A + 24B)\cos 6t + (4B - 24A)\sin 6t = 10\sin 6t$$

Compairing coefficients, we have.

$$\begin{cases} 4A + 24B = 0 \\ 4B - 24A = 10 \end{cases} \Rightarrow \begin{cases} A + 6B = 0 \\ 2B - 12A = 5 \end{cases} \Rightarrow \begin{cases} A = -\frac{15}{37} \\ B = \frac{5}{74} \end{cases}$$

Thus $x_p = -\frac{15}{37}\cos 6t + \frac{5}{74}\sin 6t$.

So the general solution to the given eqn is

$$x(t) = x_c + x_p = C_1 e^{-2t}\cos 6t + C_2 e^{-2t}\sin 6t - \frac{15}{37}\cos 6t + \frac{5}{74}\sin 6t.$$

As $x(0) = 0$, $x(0) = C_1 - \frac{15}{37} = 0 \Rightarrow C_1 = \frac{15}{37}$

$$x'(t) = \frac{15}{37}\cdot(-2)e^{-2t}\cos 6t - 6\cdot\frac{15}{37}\cdot e^{-2t}\sin 6t - 2C_2e^{-2t}\sin 6t$$
$$+ 6C_2e^{-2t}\cos 6t + \frac{15}{37}\cdot 6\sin 6t + \frac{5}{74}\cdot 6\cos 6t$$

$$x'(0) = \frac{15}{37}\cdot(-2) + 6C_2 + \frac{5}{74}\cdot 6 = 0 \Rightarrow C_2 = \left(\frac{30}{37} - \overset{\frac{15}{37}}{\cancel{\frac{30}{74}}}\right)/6$$

$$\Rightarrow C_2 = \frac{15}{37} \cdot \frac{1}{6} = \frac{5}{74}$$

Thus we have the solution to the initial value problem

$$x(t) = -\frac{15}{37}\cos 6t + \frac{5}{74}\sin 6t + \frac{15}{37}e^{-2t}\cos 6t + \frac{5}{74}e^{-2t}\sin 6t$$

Remark: In your WebWork, you might neet to convert the fractions to digits if you keep getting an error. For example, the above solution can be typed as

x(t) = e^(-2*t)[*0.405405*cos(6*t)+0.0675674*sin(6*t)]+-0.405405*cos(6*t)+0.0675676*sin(6*t)

(2) As $e^{-2t} \to 0$ as $t \to \infty$, we have

$$\lim_{t\to\infty} x(t) = x_p = -\frac{15}{37}\cos 6t + \frac{5}{74}\sin 6t$$

or

$\lim_{t\to\infty} x(t) =$ -0.405405*cos(6*t)+0.0675676*sin(6*t)

Exercises Related to WebWork

Exercise 3 and 4 are related to the mass-spring-dashpot system. Exercise 5 and 6 are related to acceleration-velocity models.

Exercise 3.

A spring with a 5-kg mass and a damping constant 1 can be held stretched 1 meters beyond its natural length by a force of 3 newtons. Suppose the spring is stretched 2 meters beyond its natural length and then released with zero velocity.

(1) In the notation of the text, what is the value $c^2 - 4mk$?

(2) Find the position of the mass, in meters, after t seconds.

ANS: (1) $F_s = -kx \Rightarrow -3 = -k \cdot 1 \Rightarrow k = 3$

$$c^2 - 4mk = 1^2 - 4 \times 5 \times 3 = 1 - 60 = -59 \quad m^2 kg^2/sec^2$$

(2) We have

$$mx'' + cx' + kx = 0, \quad x(0) = 2, \quad x'(0) = 0.$$

$$\Rightarrow \quad 5x'' + x' + 3x = 0, \quad x(0) = 2, \quad x'(0) = 0$$

Solving $5r^2 + r + 3 = 0 \Rightarrow r_{1,2} = \frac{-1 \pm \sqrt{1 - 4 \times 5 \times 3}}{2 \times 5} = \frac{-1 \pm \sqrt{-59}}{10}$

Thus $r_{1,2} = \frac{-1 \pm \sqrt{59}\, i}{10}$

So the general solution is

$$x(t) = e^{-\frac{1}{10}t} \cdot \left(C_1 \cos \frac{\sqrt{59}}{10} t + C_2 \sin \frac{\sqrt{59}}{10} t\right)$$

As $x(0) = 2 \Rightarrow x(0) = C_1 = 2$

$$x'(t) = -\frac{1}{10} e^{-\frac{1}{10}t} \left(C_1 \cos \frac{\sqrt{59}}{10} t + C_2 \sin \frac{\sqrt{59}}{10} t\right)$$

$$+ e^{-\frac{1}{10}t} \left(-\frac{\sqrt{59}}{10} C_1 \sin \frac{\sqrt{59}}{10} t + \frac{\sqrt{59}}{10} C_2 \cos \frac{\sqrt{59}}{10} t\right)$$

As $x'(0) = 0$, $x'(0) = -\frac{1}{10} \cancel{C_1}^{\,2} + \frac{\sqrt{59}}{10} C_2 = 0 \Rightarrow C_2 = \frac{10}{\sqrt{59}} \cdot \frac{1}{5} = \frac{2}{\sqrt{59}}$

Thus $x(t) = e^{-\frac{1}{10}t} \left(2 \cos \frac{\sqrt{59}}{10} t + \frac{2}{\sqrt{59}} \sin \frac{\sqrt{59}}{10} t\right)$

Exercise 4.

Suppose a spring with spring constant $4\ \mathrm{N/m}$ is horizontal and has one end attached to a wall and the other end attached to a $3\ \mathrm{kg}$ mass. Suppose that the friction of the mass with the floor (i.e., the damping constant) is $1\ \mathrm{N \cdot s/m}$.

(1) Set up a differential equation that describes this system.

(2) Find the general solution to your differential equation from the previous part.

(3) Is this system under damped, over damped, or critically damped?

(4) What is the value of the damping constant that would make the system critically damped?

(1) $3x'' + x' + 4x = 0$

(2) We have char. eqn

$$3r^2 + r + 4 = 0 \Rightarrow r_{1,2} = \frac{-1 \pm \sqrt{1-48}}{6} = \frac{-1 \pm \sqrt{47}\,i}{6}$$

Thus $x(t) = e^{-\frac{t}{6}}\left(C_1 \cos\left(\frac{\sqrt{47}}{6}t\right) + C_2 \sin\left(\frac{\sqrt{47}}{6}t\right)\right)$

(3). As we have two complex solutions to the char. eqn, we know the system is under damped

(4) Assume for $c > 0$ the system

$$3x'' + cx' + 4x = 0$$

is critically damped. We need $\Delta = c^2 - 4 \times 3 \times 4 = 0$

$\Rightarrow c = 4\sqrt{3}$

Exercise 5.

A car traveling at $40 \mathrm{ft} / \mathrm{sec}$ decelerates at a constant 2 feet per second squared. How many feet does the car travel before coming to a complete stop?

Solution. Let $s(t)$ be the distance covered by the car t seconds after stepping on the brakes. Suppose the car decelerates at g feet per second squared. Then

$$s''(t)=-g$$

and

$$s'(t)=-gt+v_0$$

where v_0 is the speed of the car at time 0 . Integrating again gives

$$s(t)=-\frac{gt^2}{2}+v_0 t.$$

The integration constant in this case is 0 since at time $t=0$ the car has covered a distance of 0 feet. We are asking how far the car travels until it comes to a stop. At that time the speed is 0 , which gives

$$t=\frac{v_0}{g}.$$

Substituting this time into the distance formula gives

$$s=-\frac{g v_0^2}{2 g^2}+\frac{v_0^2}{g}=-\frac{v_0^2}{2 g}+\frac{v_0^2}{g}=\frac{v_0^2}{2 g}.$$

Substituting

$$v_0=40 \mathrm{ft} / \mathrm{sec}, \quad g=2 \mathrm{ft} / \mathrm{sec}^2$$

gives the answer:

$$s \approx 400.00 \text { feet } .$$

Exercise 6.

A ball is shot straight up into the air with initial velocity of $50 \mathrm{ft} / \mathrm{sec}$. Assuming that the air resistance can be ignored, how high does it go? (Assume that the acceleration due to gravity is $32 \mathrm{ft}$ per second squared.)

Solution.

We have

$$\frac{dv}{dt} = -g$$

where g is the acceleration due to gravity ($32 \mathrm{ft} / \mathrm{s}^2$, downward).

So we have

$$v(t) = -gt + C$$

We know that the initial velocity of the ball $v(0)$ is $50 \mathrm{ft} / \mathrm{s}$ (upwards), so we can use this to solve for the constant of integration C :

$$50 = -0 + C \Rightarrow C = 50$$

Thus, the velocity of the ball as a function of time is:

$$v(t) = -32t + 50$$

The ball reaches its highest point when its velocity is 0 (it momentarily stops moving up before starting to fall down). We set $v(t) = 0$ and solve for t :

$$-32t + 50 = 0$$

Solving this we know the ball reaches its highest point at $t = 25/16$ seconds after it is shot.

To find the maximum height reached by the ball, we integrate the velocity function to get the position function $s(t)$:

$$s(t) = \int v(t) dt = \int (-32t + 50) dt = -16t^2 + 50t + C$$

We know that at $t = 0$, the ball is at the initial position $s(0) = 0$ (assuming it is shot from ground level), which allows us to solve for C :

$$0 = -16(0)^2 + 50(0) + C \Rightarrow C = 0$$

So, the position function is:

$$s(t) = -16t^2 + 50t$$

We have $s(25/16) \approx 39.0625$.